

Higher order terms in an improved heterotic M -theory

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ABSTRACT: Curvature-squared terms are added to a consistent formulation of supergravity on manifolds with boundary which is meant to represent the low energy limit of strongly coupled heterotic string theory. These terms are necessary for the cancellation of gravitational anomalies and for reductions to lower dimensions with broken chiral symmetry. The consequences of anomaly cancellation when flux and extrinsic curvature terms are taken into account have yet to be fully exploited, but some implications for flux terms are discussed here.

KEYWORDS: Supergravity Models, M-Theory.

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1. Introduction

Some time ago, Horava and Witten [1, 2] proposed that the low energy limit of strongly coupled heterotic string theory could be formulated as 11-dimensional supergravity on a manifold with boundary. This opened up the possibility that matter might exist on a surface embedded in the 11-dimensional spacetime with supergravity taking care of the gravitational interactions. Although the theory has received less attention recently than type IIB superstring theory, it nevertheless remains a possible starting point for particle phenomenology [3, 4].

The original formulation of Horava and Witten contained some serious problems which limited the range of validity of the 11-dimensional limit. These problems were solved recently using a new formulation of supergravity on manifolds with boundary [5–7]. The most serious problem affecting the model was that it was expressed as a series in the factor $\kappa_{11}^{2/3}$ multiplying the matter action, which worked well at leading and next-to-leading order but became ill-defined thereafter. This problem was resolved by a simple modification to the boundary conditions resulting in a low energy theory which is supersymmetric to all orders in $\kappa_{11}^{2/3}$.

The aim of the present paper is to add curvature-squared terms to the new formulation of supergravity on manifolds with boundary. These terms are necessary for the cancellation of gravitational anomalies [2, 8, 7], and they are important for reductions to lower dimensions with broken chiral symmetry [3, 4, 9–12]. Higher order terms should therefore be present if the theory is truly the low energy limit of the strongly coupled heterotic string. Curvature-squared terms have been included in the boundary action, for example by Lukas et al. [9], but they have never been shown to be part of a supersymmetric theory before.

The methodology adopted will be to construct the boundary conditions and the action of the theory order by order in derivatives, imposing the local symmetries at each

stage. Anomaly cancellation will be brought about by the Green-Schwarz mechanism [13], modified to accommodate boundaries [2, 7]. The results contain all terms with up to five derivatives and two fermi fields. A remarkable feature is that the action to this order is uniquely determined, with only one free parameter κ_{11} . It seems likely that this determinism in the theory will occur at higher orders in the curvatures, leaving no room for free parameters apart from the gravitational coupling.

Before proceeding, it will be helpful to repeat some of the ingredients of the improved version of low-energy heterotic M -theory described in ref. [6]. The theory is formulated on a manifold \mathcal{M} with a boundary consisting of two disconnected components $\partial\mathcal{M}_1$ and $\partial\mathcal{M}_2$ with identical topology. The eleven-dimensional part of the action is the conventional action for supergravity, with metric g_{IJ} , gravitino ψ_I and antisymmetric tensor C_{IJK} [14]. The boundary terms which make the supergravity action supersymmetric are [15],

$$S_0 = \frac{1}{\kappa_{11}^2} \int_{\partial\mathcal{M}} \left(\hat{K} \mp \frac{1}{4} \bar{\psi}_A \Gamma^A \Gamma^B \psi_B \right) dv, \tag{1.1}$$

where K is the extrinsic curvature of the boundary and A, B, \dots denote tangential indices. Hats denote the standardised subtraction of gravitino terms to make a supercovariant expression. We shall take the upper sign on the boundary component $\partial\mathcal{M}_1$ and the lower sign on the boundary component $\partial\mathcal{M}_2$.

There are additional boundary terms with Yang-Mills multiplets, scaled by a parameter ϵ ,

$$S_{\text{YM}} = -\frac{\epsilon}{\kappa_{11}^2} \int_{\partial\mathcal{M}} dv \left(\frac{1}{4} \text{tr} F^2 + \frac{1}{2} \text{tr} \bar{\chi} \Gamma^A D_A (\hat{\Omega}^{**}) \chi + \frac{1}{4} \bar{\psi}_A \Gamma^{BC} \Gamma^A \text{tr} F_{BC}^* \chi \right), \tag{1.2}$$

where $F^* = (F + \hat{F})/2$ and the connection $\Omega^{**} = (\Omega + \Omega^*)/2$. The original formulation of Horava and Witten contained an extra ‘ $\chi\chi\chi\psi$ ’ term, but it is not present in the new version. The formulation given in ref. [7] was only valid to order R , and our aim here is to extend the theory to include R^2 terms and beyond.

The specification of the theory is completed by boundary conditions. For the tangential anti-symmetric tensor components,

$$C_{ABC} = \mp \frac{\sqrt{2}}{12} \epsilon \left(\omega_{ABC}^Y \mp \omega_{ABC}^X \right). \tag{1.3}$$

where ω^Y is the Yang-Mills Chern-Simons form and ω^X is a bilinear gaugino term. These boundary conditions replace the modified Bianchi identity in the old formulation. A suggestion along these lines was made in the original paper of Horava and Witten [2]. For the gravitino,

$$\Gamma^{AB} (P_{\pm} + \epsilon \Gamma P_{\mp}) \psi_A = \epsilon J_Y^A, \tag{1.4}$$

where P_{\pm} are chiral projectors using the outward-going normals, Γ is a bilinear gaugino term and J_Y is the Yang-Mills supercurrent. The resulting theory is supersymmetric *to all orders* in the parameter ϵ , but the gauge anomalies only vanish if the gauge groups on the boundaries are both E_8 and

$$\epsilon = \frac{1}{4\pi} \left(\frac{\kappa_{11}}{4\pi} \right)^{2/3}. \tag{1.5}$$

Further details of the anomaly cancellation, and additional Green-Schwarz terms, can be found in ref. [7].

The gravitational anomaly vanishes if we introduce an extra term into eq. (1.3) involving the Chern-Simons term ω^L for local Lorentz transformations. The calculations which follow can be seen as an attempt to find the supersymmetric completion of the new boundary conditions with the local Lorentz term. These boundary conditions are sufficient to determine the boundary action. Section 2 lays down the general strategy and sets up the derivative expansion scheme. Section 3 gives results up to fifth order in derivatives for the boundary terms in the action and for the boundary conditions. The last part of section 3 considers anomaly cancellation and discusses the generalisation of the earlier results to all orders in the curvature. The results are collected together in the conclusion.

The conventions used follow Weinberg [16]. The metric signature is $- + \dots +$. The gamma matrices satisfy $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$ and $\Gamma^{I\dots K} = \Gamma^{[I} \dots \Gamma^{K]}$. Eleven dimensional vector indices are denoted by I, J, \dots . The coordinate indices on the boundary are denoted by A, B, \dots , tetrad ones by \hat{A}, \hat{B}, \dots and the (outward unit) normal direction by N .

2. Supersymmetry transformations

Construction of the higher order terms is based on the ingenious method introduced by Bergshoeff et al. [18, 17]. We combine the spin connection and gravitino derivatives into a pair $\{\omega_{ABC}^-, \psi_{AB}\}$ which is almost a Yang-Mills multiplet. Adding the higher order terms is similar to adding Yang-Mills multiplets, which we know how to do. Unfortunately, in 11 dimensions, normal components and flux terms complicate the simple picture and enhance the technical difficulties.

We shall start from the transformation rules for the graviton multiplet and devise a consistent derivative expansion scheme. Then we shall construct quantities which are optimised to make the best possible Yang-Mills multiplet. In the next section we construct the boundary terms in the action to fifth order in derivatives. The following section extends the boundary conditions to fifth order in derivatives and confirms that they are supersymmetric

We shall use the parameter α to keep track of the order of terms in our derivative expansion. The order of terms should be preserved by the supersymmetry transformations, which are

$$\delta e^{\hat{I}}{}_J = \frac{1}{2} \bar{\eta} \Gamma^{\hat{I}} \psi_J \tag{2.1}$$

$$\delta \psi_I = D_I(\hat{\Omega})\eta + \frac{\sqrt{2}}{288} (\Gamma_I^{JKLM} - 8\delta_I^J \Gamma^{KLM}) \eta \hat{G}_{JKLM} \tag{2.2}$$

$$\delta C_{IJK} = -\frac{\sqrt{2}}{8} \bar{\eta} \Gamma_{[IJ} \psi_{K]} \tag{2.3}$$

where G is the abelian flux tensor. We also require that $C_{ABC} \sim \omega_{ABC}^L$ on the boundary.

The ordering we shall use is,

$$R_{ABCD} = O(\alpha^2) \quad \psi_A = O(\alpha) \quad (2.4)$$

$$G_{NABC} = O(\alpha^2) \quad D_{[A}\psi_{B]} = O(\alpha^2) \quad (2.5)$$

$$G_{ABCD} = O(\alpha^4) \quad D_{[N}\psi_{B]} = O(\alpha^3) \quad (2.6)$$

Additional tangential derivatives increase the order by one. This expansion scheme is consistent with the Calabi-Yau reductions found in the literature [9], where the small parameter is related to the curvature of the Calabi-Yau space.

The first quantity we construct is the gravitino curvature ψ_{AB} . We start from the supersymmetry transformation of the tangential gravitino from eq. (2.2) to two-fermi order, which can be written,

$$\delta\psi_A = \mathcal{D}_A\eta = (D_A + \mathcal{A}_A)\eta \quad (2.7)$$

where D_A uses the Levi-Civita connection and \mathcal{A}_A contains a combination of abelian-flux terms and gamma matrices. The analogue of the curvature is defined by

$$[\mathcal{D}_A, \mathcal{D}_B] = \mathcal{R}_{AB}. \quad (2.8)$$

The quantity \mathcal{R}_{AB} is a tensor which takes values in the gamma-matrix algebra,

$$\mathcal{R}_{AB} = -\frac{1}{4}R_{ABIJ}\Gamma^{IJ} + 2\mathcal{D}_{[A}\mathcal{A}_{B]} - [\mathcal{A}_A, \mathcal{A}_B]. \quad (2.9)$$

The new derivative is used to define the gravitino curvature ψ_{AB} ,

$$\psi_{NA} = 2\mathcal{D}_{[N}\psi_{A]} \quad (2.10)$$

$$\psi_{AB} = 2\mathcal{D}_{[A}\psi_{B]} - \Gamma_{[A}\psi_{B]}N. \quad (2.11)$$

It may help understand this construction to recall that, in the reduction of 11-dimensional supergravity, the 10-dimensional gravitino is $\psi_A + \Gamma_A\psi_N/2$. If the normal derivatives vanish, then ψ_{AB} is the usual 10-dimensional gravitino curvature. The supersymmetry transformation of the gravitino curvature is

$$\delta\psi_{AB} = \mathcal{R}_{AB}\eta - 2\Gamma_{[A}\mathcal{R}_{B]}N\eta. \quad (2.12)$$

Note that R_{NABC} is very small due to Gauss-Codacci relations, of order α^5 , and this variation basically depends on R_{ABCD} plus abelian-flux terms.

The supersymmetry transformations are only required on the boundary, where it proves convenient to decompose the flux-gamma-matrix combinations into tangential and normal components,

$$X = \frac{\sqrt{2}}{72}G_{NABC}\Gamma^{ABC}, \quad X_A = \frac{\sqrt{2}}{8}G_{NABC}\Gamma^{BC}, \quad (2.13)$$

$$Y = \frac{\sqrt{2}}{288}G_{ABCD}\Gamma^{ABCD}, \quad Y_A = \frac{\sqrt{2}}{24}G_{ABCD}\Gamma^{BCD}, \quad Y_{AB} = \frac{\sqrt{2}}{8}G_{ABCD}\Gamma^{CD}. \quad (2.14)$$

For example,

$$\mathcal{A}_A = \Gamma_A(\Gamma_N X + Y) - \Gamma_N X_A - Y_A, \quad (2.15)$$

$$\mathcal{A}_N = -2X + \Gamma_N Y. \quad (2.16)$$

The supersymmetry parameter is chiral on the boundaries, with $\Gamma_N \eta = \mp \eta$ depending on which boundary we choose. The results below take $\Gamma_N \eta = -\eta$.

Next, we turn to the Levi-Civita spin connection $\omega_{A\hat{B}\hat{C}}$. This does not transform like a Yang-Mills gauge field, but we can adapt an idea from 10 dimensions [17] and try the addition of a G -flux term,

$$\omega_{A\hat{B}\hat{C}}^- = \omega_{A\hat{B}\hat{C}} + \frac{1}{\sqrt{2}} G_{NA\hat{B}\hat{C}}. \quad (2.17)$$

The transformation rules for the pair $\hat{\omega}_{A\hat{B}\hat{C}}^-$ and ψ_{AB} become

$$\delta \hat{\omega}_{A\hat{B}\hat{C}}^- = -\frac{1}{2} \bar{\eta} \Gamma_A \psi_{\hat{B}\hat{C}} + y_{A\hat{B}\hat{C}} \quad (2.18)$$

$$\delta \psi_{AB} = -\frac{1}{4} R^-_{CDAB} \Gamma^{CD} \eta + y_{AB} \eta, \quad (2.19)$$

where the minus superscript on the curvature indicates use of the ω^- connection. The leading terms are $O(\alpha^2)$, whereas $y_{A\hat{B}\hat{C}}$ and y_{AB} are both $O(\alpha^3)$. These correction terms are given by

$$y_{A\hat{B}\hat{C}} = -\frac{1}{4} \bar{\eta} \Gamma_{\hat{B}\hat{C}} \psi_{NA} - \frac{1}{2} \bar{\eta} \{ \Gamma_{\hat{B}\hat{C}}, X \} \psi_A + \frac{1}{2} \bar{\eta} e_{A\hat{B}} \psi_{N\hat{C}} + O(\alpha^5), \quad (2.20)$$

$$y_{AB} = \Gamma_{AB} Y' - \Gamma_{[A} Y'_{B]} - Y'_{AB} + O(\alpha^5). \quad (2.21)$$

where Y' is shorthand notation for $D_N Y$. Note that $R^-_{ABCD} \neq R^-_{CDAB}$ now that the connection is no longer a metric connection.

In order to complete the set of transformation rules, we also need

$$\delta \psi_{NA} = y_{NA} \eta, \quad (2.22)$$

where

$$y_{NA} = 2D_A X + \Gamma_A Y' - Y'_A + O(\alpha^5). \quad (2.23)$$

We have not made any modification to the basic supersymmetry rules, and none appears to be necessary to the order at which we are working. All of the approximations used in this section can be replaced by exact expressions, but the approximate ones are sufficient for the subsequent sections.

3. Higher order terms

3.1 Boundary terms in the action

At leading order in α , the pair $\{ \hat{\omega}_{A\hat{B}\hat{C}}^-, \psi_{AB} \}$ form a Yang-Mills multiplet and we can add this to the boundary conditions and the action in the same way as the existing Yang-Mills

multiplet described in the introduction. We use a new coupling ϵ^L for the new multiplet, and anomaly cancellation fixes ϵ^L [7],

$$\epsilon^L = -\frac{1}{2}\epsilon. \quad (3.1)$$

The boundary conditions can be read off eqs. (1.3) and (1.4),

$$C_{ABC} = -\frac{\sqrt{2}}{12}\epsilon(\omega_{ABC}^Y + \omega_{ABC}^X) + \frac{\sqrt{2}}{24}\epsilon(\omega_{ABC}^L + \omega_{ABC}^\psi), \quad (3.2)$$

$$\Gamma^{AB}P_+\psi_B = \epsilon J_Y^A - \frac{1}{2}\epsilon J_L^A, \quad (3.3)$$

$$\hat{K}_{AB} - \frac{1}{2}g_{AB}\hat{K} = \epsilon T_{AB}^Y - \frac{1}{2}\epsilon T_{AB}^L, \quad (3.4)$$

where T_{AB}^Y is the Yang-Mills stress tensor and

$$\omega_{ABC}^\psi = \frac{1}{4}\bar{\psi}_{DE}\Gamma_{ABC}\psi^{DE}, \quad (3.5)$$

$$J_L^A = \frac{1}{4}\Gamma^{BC}\Gamma^A R_{BCDE}\psi^{DE}, \quad (3.6)$$

$$T_{AB}^L = R^-{}_{ACDE}R^-{}_{B}{}^{CDE} - \frac{1}{4}g_{AB}R^-{}_{CDEF}R^-{}^{CDEF} + \psi\text{-terms}. \quad (3.7)$$

The Yang-Mills results imply that eq. (3.4) is a necessary and sufficient condition for supersymmetry of the boundary conditions (3.2) and (3.3).

There are also new curvature-squared terms in the boundary action, which we obtain from the Yang-Mills terms in eq. (1.2),

$$S_{RR} = \frac{\epsilon}{2\kappa_{11}^2} \int_{\partial\mathcal{M}} dv \left(\frac{1}{4}R^-{}_{ABCD}R^-{}^{ABCD} + \frac{1}{2}\bar{\psi}_{BC}\Gamma^A D_A(\omega)\psi^{BC} + \frac{1}{4}\bar{\psi}_A\Gamma^{BC}\Gamma^A R_{BCDE}\psi^{DE} \right). \quad (3.8)$$

The supersymmetry of the full action with the new boundary terms follows from the gravity-Yang-Mills calculation given previously [6]. The modified curvature $R^-{}_{ABCD}$ has been used for consistency between the derivative orders of the bosonic and fermionic terms. Note that terms involving the square of the Ricci tensor only appear at order α^8 in the ordering scheme being used.

Another important property of the full action is that it should be stationary under variations of the fields about solutions to the field equations with the specified boundary conditions. Variations of the new boundary term with the tetrad can be decomposed into metric variations and local Lorentz rotations (see appendix B in [6]),

$$\delta S_{RR} = \frac{\epsilon}{2\kappa_{11}^2} \int_{\partial\mathcal{M}} dv \left\{ \delta\omega^-{}_{B\hat{C}\hat{D}} \left(D_A R^{AB\hat{C}\hat{D}} - \frac{1}{2}D_E(\bar{\psi}_A\Gamma^{BE}\Gamma^A\psi^{\hat{C}\hat{D}}) \right) - \frac{1}{2}\delta g_{AB}T^{LAB} \right\}. \quad (3.9)$$

The surface stress-tensor term is $O(\alpha^4)$ in the derivative expansion and combines with the variation of the supergravity action S_{SG} to produce the boundary condition eq. (3.4). The

metric variation provides a good way to determine the fermion terms in the stress-tensor.¹ The Bianchi identity and the gravitino field equation together imply that the variation of the surface connection gives no contribution to δS_{RR} at leading order in the derivative expansion.

Variations of the action with the gravitino can be split up in the following way,

$$\delta S_{RR} = \frac{\epsilon}{2\kappa_{11}^2} \int_{\partial\mathcal{M}} dv \left\{ \overline{\delta\psi}^{BC} \left(\Gamma^A D_A \psi_{BC} + \frac{1}{4} R_{DEBC} \Gamma^A \Gamma^{DE} \psi_A \right) + \overline{\delta\psi}_A J_L^A \right\}. \quad (3.10)$$

The supercurrent term is $O(\alpha^5)$ and contributes to the boundary condition Eq (3.3). The gravitino field equation can be used to show that the remaining terms are only $O(\alpha^6)$, and they play no role at leading order.

Finally, it is possible to reduce the 11-dimensional action to 10-dimensions to obtain the low energy limit of the weakly coupled heterotic superstring. The result of dimensional reduction agrees with the higher-order action obtained from supergravity 10-dimensions [18]. The curvature-squared terms obtained from string amplitude calculations also agree, up to allowed metric redefinitions [19].

3.2 Boundary conditions and G -fluxes

At fifth order in derivatives new G -flux terms begin to contribute to the boundary conditions. Due to the connection between the boundary conditions and the Green-Schwarz anomaly cancellation mechanism, these terms allow us to deduce some of the G -flux terms in the anomalies. The part of the supersymmetry transformation which is exactly fifth order in derivatives will be denoted by δ_5 . For the rest of this section we shall drop the Yang-Mills terms.

The fermion boundary condition can be written in the form,

$$P_+ \psi_A = -\frac{1}{24} \epsilon (\Gamma_A^{BC} - 10\delta_A^B \Gamma^C) R^-_{BCDE} \psi^{DE} + \epsilon f_A^{BC} \psi_{BC}, \quad (3.11)$$

where f_A^{BC} contains G -flux terms and gamma-matrices. When we drop the three-fermi terms, variation of the fermion boundary condition can be done using eq. (2.19) and gamma matrix identities. The fifth order supersymmetry variation $\delta_5 P_+ \psi_A$ vanishes for

$$f_A^{BC} = \delta_A^{[B} Y'^{C]} + \frac{1}{6} \Gamma_A Y'^{BC}, \quad (3.12)$$

where Y^A was defined in eq. (2.14) and prime denotes a normal derivative.

The antisymmetric tensor is a little more complex. The proposed boundary condition is that

$$C_{ABC} = \frac{\sqrt{2}}{24} \epsilon \left(\omega_{ABC}^L + \omega_{ABC}^\psi + \omega_{ABC}^G \right), \quad (3.13)$$

¹Variation of the original supergravity action makes a contribution to T_{AB}^L . This was given incorrectly in section 2 of [6]. I am grateful to Paul Saffin for pointing out this mistake.

where

$$\omega_{ABC}^L = \omega_{ABC}^L(\hat{\omega}^-) \quad (3.14)$$

$$\omega_{ABC}^\psi = \frac{1}{4}\bar{\psi}_{DE}\Gamma_{ABC}\psi^{DE} - 6\bar{\psi}_{N[A}\psi_{BC]} - 12\bar{\psi}_{[AB}X\psi_{C]} \quad (3.15)$$

$$\omega_{ABC}^G = -\frac{1}{3} * \hat{G}'_{ABC}{}^{DEF}\hat{G}_{NDEF} + 3\hat{G}_{N[A}{}^{DE}\hat{G}'_{BC]DE}. \quad (3.16)$$

The dual tensor

$$*G'_{ABCDEF} = \frac{1}{24}\varepsilon_{ABCDEF}{}^{PQRS}G'_{PQRS}, \quad (3.17)$$

where $G'_{ABCD} = D_N G_{ABCD} = -4D_{[A}G_{BCD]N}$.

Variation of the antisymmetric tensor field on the boundary using the bulk supersymmetry transformations gives,

$$\delta C_{ABC} = -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[AB}\psi_{C]}. \quad (3.18)$$

Since $\eta = P_-\eta$, we can replace ψ_A by $P_+\psi_A$ and use the gravitino boundary condition (3.11) to get the fifth order transformation,

$$\delta_5 C_{ABC} = -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[AB}f_{C]}{}^{DE}\psi_{DE}. \quad (3.19)$$

Variation of the terms on the right hand side of the boundary condition using eqs. (2.18-2.22) gives

$$\delta_5 \omega_{ABC}^L = -6\bar{\eta}R_{[BC}\psi_{NA]} - 12\bar{\eta}\{R_{[AB}, X\}\psi_{C]}, \quad (3.20)$$

$$\delta_5 \omega_{ABC}^\psi = 6\bar{\eta}R_{[BC}\psi_{NA]} - 12\bar{\psi}_{[AB}X D_{C]}\eta \quad (3.21)$$

$$+ 12\bar{\eta}R_{[AB}X\psi_{C]} + \frac{1}{2}\bar{\eta}y^{DE}\Gamma_{ABC}\psi_{DE} - 6\bar{\eta}y_{NA}\psi_{BC},$$

$$\delta_5 \omega_{ABC}^G = -\frac{1}{4}\sqrt{2} * G'_{ABC}{}^{DEF}\bar{\eta}\Gamma_F\psi_{DE} + \frac{9}{2}\sqrt{2}G'_{AB}{}^{DE}\bar{\eta}\Gamma_{[C}\psi_{DE]}. \quad (3.22)$$

The best way to deal with the $D_A\eta$ contribution is to remove a total derivative,

$$\delta_5 \omega^L + \delta_3 \omega^\psi = -12D_{[A}(\bar{\eta}X\psi_{BC])} + 12\bar{\eta}(D_{[A}X)\psi_{BC]} + \frac{1}{2}\bar{\eta}y^{DE}\Gamma_{ABC}\psi_{DE} - 6\bar{\eta}y_{NA}\psi_{BC}, \quad (3.23)$$

where use has been made of the identity

$$D_{[A}\psi_{BC]} = R_{[AB}\psi_{C]}. \quad (3.24)$$

We can absorb the total derivative into an abelian transformation of the C field. After difficult gamma-matrix manipulations,

$$\delta_5 \omega^L + \delta_3 \omega^\psi = -3\bar{\eta}\Gamma_{[AB}f_{C]}{}^{DE}\psi_{DE} + \frac{1}{4}\sqrt{2} * G'_{ABC}{}^{DEF}\bar{\eta}\Gamma_F\psi_{DE} - \frac{9}{2}\sqrt{2}G'_{AB}{}^{DE}\bar{\eta}\Gamma_{[C}\psi_{DE]}. \quad (3.25)$$

The last two terms cancel with eq. (3.22), leaving a term which matches eq. (3.19). We can conclude that the boundary condition on C is supersymmetric.

3.3 Anomaly cancellation

Earlier in this section we used the fact that anomaly cancellation requires the combination of Chern-Simons forms

$$\omega^Y - \frac{1}{2}\omega^L. \tag{3.26}$$

in the boundary condition for the antisymmetric field. This combination originates in the 12-form I_{12} which generates the gauge, gravity and supergravity anomalies. Horava and Witten obtained an expression for this 12-form by combining gaugino and gravitino contributions,

$$I_{12} = \frac{1}{12(2\pi)^5}(I_4^3 - 4I_4X_8), \tag{3.27}$$

where

$$I_4 = \text{tr} F^2 - \frac{1}{2}\text{tr} R^2 \tag{3.28}$$

$$X_8 = -\frac{1}{8}\text{tr} R^4 + \frac{1}{32}(\text{tr} R^2)^2. \tag{3.29}$$

The usual notation convention is used now where exterior products are implied rather than explicit. The combination (3.26) allows the gauge variation of the CGG term in the action to cancel the anomalies descended from I_4^3 .

We have found that supersymmetry demands G -flux terms to appear in addition to the Chern-Simons terms in the boundary conditions. Anomaly cancellation will only occur if these terms also appear in I_4 ,

$$I_4 = \text{tr} F^2 - \frac{1}{2}\text{tr} R^{-2} + d\omega^G + O(\alpha^6), \tag{3.30}$$

where ω^G was given in eq. (3.16). Note that we can only determine the G_{NABC} terms at this order, even though G_{ABCD} terms may also contribute to I_4 . As a matter of fact, both G -flux and extrinsic curvature terms can contribute to the anomaly, since these were both dropped from the original anomaly calculations.

The anomaly 12-form has now been calculated with G -flux terms by Lukic and Moore [20]. Unfortunately, a direct comparison is complicated for a number of reasons. Firstly, Lukic and Moore include a ‘ $G\chi\chi$ ’ term in their boundary action as suggested by Horava and Witten [2], but which is not allowed in the improved theory. Secondly, the fields in the direct anomaly calculation satisfy the background field equations. The boundary conditions only give the I_4^3 part of the anomaly and we need the full expression to compare when subject to field equations.

For the remainder of this section we turn from the G -flux terms to higher order curvature terms. The I_4X_8 term in I_{12} can be cancelled by a Green-Schwarz term CX_8 in the 11-dimensional action [8]. To make sense of this term, X_8 has to be defined in the 11-dimensional bulk so that it reduces to (3.29) on the boundary. There is no need to modify the boundary condition on the C field on account of the CX_8 term, the boundary condition being determined only by I_4 as long as the anomaly takes the general form (3.27) so that the Green Schwarz mechanism can be applied.

We have seen already how anomaly cancellation leads to a unique combination of curvature-squared terms in the boundary action. This occurs also at higher orders in curvature. The boundary condition on the C field is determined by I_4 . The other boundary conditions are then fixed by supersymmetry. In turn the boundary action, which is determined by the boundary conditions, must also be fixed by I_4 . Terms which are higher order than the square of the curvature can arise in this way from extrinsic curvature contributions to the anomaly. These can be replaced by higher order intrinsic curvature terms by using the boundary condition on the extrinsic curvature.

As an example, we could consider the ‘ $\epsilon^2 R^4$ ’ interaction terms in the supergravity action which are related to the ‘ CX_8 ’ term by supersymmetry [8]. These will bring in boundary terms of the form ‘ $\epsilon^2 KR^3$ ’, equivalent to ‘ $\epsilon K^2 R$ ’ after we apply the boundary condition on the extrinsic curvature K . Anomaly cancellation will now require another modification to I_4 , introducing ‘ $K^2 R$ ’ terms. In principle, we could reconstruct the extrinsic curvature terms in the anomaly term this way. Comparing these against a direct calculation of the anomaly would be a highly non-trivial consistency check.

Another reason this approach may be of interest is that the arguments made so far for the low energy effective action can be applied equally well to the quantum field theory effective action for supergravity on a manifold with boundary. By including a boundary we introduce anomalies. If these can be cancelled by a Green-Schwarz type of mechanism, then local terms in the boundary action are severely constrained. In particular, any supersymmetric counterterms to the theory which required boundary contributions to the action would not be allowed.

4. Conclusion

11-dimensional supergravity on a manifold with boundary shows an amazing robustness. At each successive order in derivatives, the anomaly-free extension of the theory is very tightly constrained, but so far this has not forced any internal contradictions. This is consistent with the idea that the construction produces the low-energy limit of a well-defined theory of some kind.

The results for the higher order terms obtained in this paper can be summarised as follows. First of all, the curvature-squared terms in the boundary of the supergravity action up to fifth order in derivatives are

$$S_{RR} = \frac{\epsilon}{2\kappa_{11}^2} \int_{\partial\mathcal{M}} dv \left(\frac{1}{4} R^-{}_{ABCD} R^{-ABCD} + \frac{1}{2} \bar{\psi}_{BC} \Gamma^A D_A(\omega) \psi^{BC} + \frac{1}{4} \bar{\psi}_A \Gamma^{BC} \Gamma^A R_{BCDE} \psi^{DE} \right), \tag{4.1}$$

where ψ_{AB} is the gravitino curvature eq. (2.11) and the minus superscript indicates use of the modified Lorentz connection with G -flux terms eq. (2.17). The theory now has vanishing gravity and supergravity anomalies, as well as the vanishing gauge anomaly which existed previously. The supersymmetric boundary conditions up to fifth order in derivatives and

two fermi fields are,

$$C_{ABC} = \frac{\sqrt{2}}{24}\epsilon \left(\omega_{ABC}^L + \omega_{ABC}^\psi + \omega_{ABC}^G \right) \quad (4.2)$$

$$P_+\psi_A = -\frac{1}{24}\epsilon \left(\Gamma_A^{BC} - 10\delta_A^{B\Gamma C} \right) R^-_{BCDE}\psi^{DE} + \epsilon f_A^{BC}\psi_{BC} \quad (4.3)$$

$$\hat{K}_{AB} = -\frac{1}{2}\epsilon \left(R^-_{ACDE}R^-_B{}^{CDE} - \frac{1}{12}g_{AB}R^-_{CDEF}R^{-CDEF} \right) + \psi\text{-terms.} \quad (4.4)$$

where the ψ -terms in eq. (4.4) can be obtained by variation of the full action whilst keeping the surface connection fixed, and

$$\omega_{ABC}^L = \omega_{ABC}^L(\hat{\omega}^-) \quad (4.5)$$

$$\omega_{ABC}^\psi = \frac{1}{4}\bar{\psi}_{DE}\Gamma_{ABC}\psi^{DE} - 6\bar{\psi}_{N[A}\psi_{BC]} - \frac{\sqrt{2}}{6}G_{NPQR}\bar{\psi}_{[AB}\Gamma^{PQR}\psi_{C]} \quad (4.6)$$

$$\omega_{ABC}^G = -\frac{1}{3} * \hat{G}'_{ABC}{}^{DEF}\hat{G}_{NDEF} + 3\hat{G}_{N[A}{}^{DE}\hat{G}'_{BC]DE} \quad (4.7)$$

$$f_A{}^{BC} = \frac{\sqrt{2}}{24}\Gamma^{PQR}\delta_A{}^{[B}G'^{C]}{}_{PQR} + \frac{\sqrt{2}}{48}\Gamma_A\Gamma^{PQ}G'_{PQ}{}^{BC}. \quad (4.8)$$

Hats denote the supercovariant quantity constructed by adding fermion terms and prime denotes a derivative in the normal direction.

The flux terms imply new contributions to the gravitino anomaly of supergravity on a manifold with boundary. Some progress has been made in calculating these terms directly [20], but so far more work is needed for a full comparison to be made. A direct calculation of the gravitational anomaly including flux terms and extrinsic curvatures would give an important check that the Green-Schwarz mechanism can be used at higher derivative orders to obtain an anomaly-free theory.

Introducing the boundary means that total divergences which are usually discarded when discussing supersymmetry have to be retained. These total divergences are particularly dangerous when they start to interfere with the anomaly cancellation mechanism, as described in section III C. This restricts the addition of new bulk interaction terms to the supergravity action [21]. There may also be important implications for the allowed counterterms in quantised 11-dimensional supergravity [22], and it would be interesting to examine both interaction and counterterms on a manifold with boundary.

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